I. J. SCHOENBERG

The Landau problem I.
The case of motions on sets

Dedicated to N. G. de Bruijn on the occasion of his 60th birthday

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Department of Mathematics, United States Military Academy, West Point, New York 10996

1. STATEMENT OF THE PROBLEM

Let $f(t)$ be real-valued, $-\infty<t<\infty$. As we think of t as time, we denote its derivatives by $\dot{f}(t)$ and $\ddot{f}(t)$. We assume that $f\in C^1(\mathbb{R})$, hence $\dot{f}\in C(\mathbb{R})$, and that $\ddot{f}(t)$ is piecewise continuous with dicontinuities only of the first kind. Using the supremum norm on $\mathbb{R}$, we have the well known inequality of Landau

(1) $\quad \|\dot{f}\|\leqq\sqrt{2\cdot\|f\|\cdot\|\ddot{f}\|}$, (See [1], [2]).

If $a>0$ and $f(t)$ satisfies the inequalities

(2) $\quad -a\leqq f(t)\leqq a$ for all t,

then (1) shows that

(3) $\quad \|\dot{f}\|\leqq\sqrt{2a}\sqrt{\|\ddot{f}\|}$.

Here $\sqrt{2a}$ is the best constant, as shown by the special motion $\bar{f}(t)=4a(t-t^2)$ if $0\leqq t\leqq 1$, which is extended to all real t by the functional equation $\bar{f}(t+1)=-\bar{f}(t)$, for all real t, with the result that $\|\bar{f}\|=a$, $\|\dot{\bar{f}}\|=4a$, $\|\ddot{\bar{f}}\|=8a$. This motion produces the equality sign in (3).

We may interpret (2) by saying that $f(t)$ describes a motion in the interval

(4) $\quad I_a=\{-a\leqq x\leqq a\}$.

In terms of the functional

$$(5) \qquad F(f) = \frac{\|\dot{f}\|}{\sqrt{\|\ddot{f}\|}}$$

we may restate the result (3) by writing that

$$(6) \qquad \sup_f F(f) = \sqrt{2a}$$

for all motions f within the interval I_a.

It seems rather natural to replace the interval (4) by a given *closed and connected set* S of the complex plane $\mathbf{C}$, or belonging to a finite-dimensional euclidean space. Accordingly, we consider the class of motions

$$(7) \qquad (S) = \{f(t);\, f(t) \in S \text{ for all } t,\ f(t) \not\equiv \text{constant}\}.$$

Notice that we have excluded the trivial case when $f(t) = \text{constant}$ for all t, when $F(f) = 0/0$ is undefined.

We can now formulate

Landau's problem for the set S. *To determine the quantity*

$$(8) \qquad L(S) = \sup_{f \in (S)} F(f).$$

We call $L(S)$ the Landau constant of the set S.

2. A few special sets S when $L(S)$ is known

In terms of (1.4) and the definition (1.8), we may restate the result (1.6) by writing that

$$(1) \qquad L(I_a) = \sqrt{2a}.$$

Here are a few trivial further examples. If $S = \{0 \leqq x < \infty\}$, then $f(t) = t^2 \in (S)$, and $\|\dot{f}\| = \infty$, $\|\ddot{f}\| = 2$. Therefore $L(S) = \infty/2 = \infty$. If $S = \mathbf{R}$, then $f(t) = t \in (S)$, and $L(S) = 1/0 = \infty$.

An obvious and fundamental property of the Landau constant is the following

Monotonicity property. If S_1 *and* S_2 *are two sets, then*

$$(2) \qquad S_1 \subset S_2 \textit{ implies that } L(S_1) \leqq L(S_2).$$

A few further non-trivial examples when $L(S)$ is known are the following. If S is the *circular disc*

$$(3) \qquad D_a = \{|z| \leqq a\},$$

then $I_a \subset D_a$ implies, by (2), that $L(I_a) \leqq L(D_a)$. As a matter of fact we have here equality, so that

$$(4) \qquad L(D_a) = \sqrt{2a}.$$

For the *circle*

(5) $\qquad C_a = \{|z| = a\}$

we will show that

(6) $\qquad L(C_a) = \sqrt{a}.$

It seems worthwhile to look at the two results (1) and (6) and compare their respective extremizing motions. Let $\mathscr{E}_2(t)$ be the so-called *quadratic Euler spline* (See [2, Section 2]) defined by

(7) $\qquad \mathscr{E}_2(t) = 1 - 4t^2$ if $-\frac{1}{2} \leqq t \leqq \frac{1}{2}$, and $\mathscr{E}_2(t+1) = -\mathscr{E}_2(t)$ for all t.

We found above that $\bar{f}(t) = a\mathscr{E}_2(t) \in (I_a)$ and $F(\bar{f}) = \sqrt{2a}$. For $S = C_a$ the extremizing motion is the *uniform rotation*

$$\bar{f}(t) = ae^{it}.$$

Indeed, here $\|\dot{\bar{f}}\| = a$, $\|\ddot{\bar{f}}\| = a$, hence $F(\bar{f}) = \sqrt{a}$, and this is the extremizing motion in view of (6).

Let now S be the *circular ring*

(8) $\qquad R_{a,b} = \{b \leqq |z| \leqq a\}, \ (0 < b < a).$

Here *we shall find that* *

(9) $\qquad L(R_{a,b}) = \sqrt{a + \sqrt{a^2 - b^2}}.$

We would expect to obtain from this the result (6) when $b = a$, and (4) when $b = 0$.

Our last special example is the *solid shell*

(10) $\qquad SoSh_{a,b} = \{b \leqq \sqrt{x^2 + y^2 + z^2} \leqq a\}, \ (0 < b < a).$

As this will yield most of the previous results, we formulate it as

THEOREM 1. *For the solid shell* (10) *we have that*

(11) $\qquad L(SoSh_{a,b}) = \sqrt{a + \sqrt{a^2 - b^2}}.$

Actually Theorem 1, with the same value for the Landau constant, holds for the solid spherical shell in n dimensions, $n \geqq 2$. Observe again, that $R_{a,b} \subset SoSh_{a,b}$, while their Landau constants are equal by (9) and (11).

Of an entirely different nature is

THEOREM 2. *If the set S is closed, bounded and convex, then*

(12) $\qquad L(S) = \sqrt{\text{diameter of } S}.$

* This was first shown in [3]. In Section 5 below we first derive the stronger relation (11) which is somewhat easier to prove than (9). Theorem 2 below was stated without proof in [3].

Clearly Theorem 2 implies as special cases the results (1) and (4). Again Theorem 2, and its proof below, remains valid, as it stands, in any finite-dimensional euclidean space.

3. A LEMMA

The lemma to be now discussed, will show that an extremizing function $\bar{f}(t)$, i.e. producing the equality $F(\bar{f})=L(S)$, may also be characterized as a solution of a problem of Optimal Control.

LEMMA 1. *If the motion*

(1) $\qquad \bar{f}(t)\in(S)$

has the

PROPERTY A. *If*

(2) $\qquad f(t)\in(S)$ *and* $\|\ddot{f}\|\leqq\|\ddot{\bar{f}}\|$

then

(3) $\qquad \|\dot{f}\|\leqq\|\dot{\bar{f}}\|,$

then $\bar{f}$ also has the

PROPERTY B.

(4) $\qquad F(\bar{f})=L(S),$

and conversely, if $\bar{f}$ enjoys the Property B, then $\bar{f}$ also has the Property A.

PROOF: 1. *Property A implies Property B.* We assume (1) and Property A and we are to establish (4). If $f\in(S)$, then for the motion $g(t)=f(t\sqrt{\|\ddot{\bar{f}}\|/\|\ddot{f}\|})$ we find that $g(t)\in(S)$ and also that

$$\|\ddot{g}\|=\|\ddot{f}\|(\|\ddot{\bar{f}}\|/\|\ddot{f}\|),\ \text{hence}\ \|\ddot{g}\|=\|\ddot{\bar{f}}\|.$$

Hence $g(t)$ satisfies the assumptions (2), and by Property A we conclude that $\|\dot{g}\|\leqq\|\dot{\bar{f}}\|$, and by the definition of $g(t)$, that

$$\|\dot{f}\|\sqrt{\|\ddot{\bar{f}}\|/\|\ddot{f}\|}\leqq\|\dot{\bar{f}}\|,\ \text{and so}\ F(f)\leqq F(\bar{f}).$$

But then

$$L(S)=\sup_{f\in(S)} F(t)\leqq F(\bar{f}).$$

From $F(\bar{f})\leqq L(S)$ we now conclude that (4) holds.

2. *Property B implies Property A.* Now we assume (1) and (4), and that f satisfies (2), and we are to show that (3) holds. From (4) we obtain

that $F(f) \leqq F(\tilde{f})$, or $\|\dot{f}\|/\sqrt{\|\ddot{f}\|} \leqq \|\dot{\tilde{f}}\|/\sqrt{\|\ddot{\tilde{f}}\|}$, whence

$$\|\dot{f}\| \leqq \|\dot{\tilde{f}}\| \sqrt{\|\ddot{f}\|/\|\ddot{\tilde{f}}\|} \leqq \|\dot{\tilde{f}}\|$$

in view of the second inequality (2), and (3) is established. Lemma 1 was stated without proof in [3].

4. A PROOF OF THEOREM 2

Let A and B be points of S such that

(1) $\qquad |B-A| = d = \text{diameter of } S, \ (A \in S,\ B \in S).$

From the relations (2.7) we find that $-1 \leqq \mathscr{E}_2(t) \leqq 1$ for all t, and that $\mathscr{E}_2(t)$ represents a to-and-fro motion on $[-1, 1]$. We now define on the segment $[A, B]$ the motion

(2) $$\tilde{f}(t) = \tfrac{1}{2}A(1-\mathscr{E}_2(t)) + \tfrac{1}{2}B(1+\mathscr{E}_2(t)) = \tfrac{1}{2}(B+A) + \tfrac{1}{2}(B-A)\mathscr{E}_2(t).$$

From $\|\dot{\mathscr{E}}_2\| = 4$, $\|\ddot{\mathscr{E}}_2\| = 8$, and (1), we find that

(3) $\qquad \|\dot{\tilde{f}}\| = 2d, \ \|\ddot{\tilde{f}}\| = 4d,$

and therefore

(4) $\qquad F(\tilde{f}) = \sqrt{d}.$

We wish to prove that $F(\tilde{f}) = L(S)$, which is the relation (4) of Property B of Lemma 1. By Lemma 1 it suffices to prove the following:

If $f \in (S)$ *such that*

(5) $\qquad \|\ddot{f}\| \leqq \|\ddot{\tilde{f}}\| = 4d,$

then we also have

(6) $\qquad \|\dot{f}\| \leqq \|\dot{\tilde{f}}\| = 2d.$

This we do as follows: We consider an arbitrary but fixed t_0 such that $\dot{f}(t_0) \neq 0$, and let L be a fixed straight line such that

(7) $\qquad$ *L is parallel to the vector $\dot{f}(t_0)$.**

Let the segment $[P, Q] \subset L$ be the orthogonal projection of S onto the line L. Clearly

(8) $\qquad |Q-P| \leqq |A-B| = d.$

Let $f_p(t)$ denote the orthogonal projection of the motion $f(t)$ onto L. By (5) we have

$$\|\ddot{f}_p\| \leqq \|\ddot{f}\| \leqq \|\ddot{\tilde{f}}\| = 4d$$

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* The reader is asked to sketch a diagram.

and by (8) and Landau's inequality (1.1) we find that

$$\|\dot{f}_p\| \leqq \sqrt{|Q-P|} \cdot \sqrt{\|\ddot{f}_p\|} \leqq \sqrt{d} \cdot \sqrt{4d} = 2d.$$

Now (7) implies that

$$|\dot{f}(t_0)| = |\dot{f}_p(t_0)| \leqq \|\dot{f}_p\| \leqq 2d.$$

and therefore $\|\dot{f}\| \leqq 2d$, since t_0 was arbitrary. This proves (6) and our proof is complete.

REMARK. Our proof shows that for Theorem 2 to be valid, the set S need not be convex. All that is needed is that (1) should hold, and that $[A, B] \subset S$. These assumptions already imply that $L(S) = L([A, B])$. It does seem curious that we can not use the greater freedom of motions in S to increase the value $\sqrt{d}$ of the Landau constant.

5. A PROOF OF THEOREM 1

We now turn to the solid shell

$$(1) \qquad S = SoSh_{a,b} = \{b \leqq \sqrt{x^2+y^2+z^2} \leqq a\}, \ (0 < b < a).$$

Just as in the case of the convex set A we identified the motion (4.2) as the motion that maximizes $F(f)$ among all motions within S, so we will now construct a motion again to be denoted by $\tilde{f}(t)$ ($t \in \mathbf{R}$), that will maximize $F(f)$ for the solid shell (1).

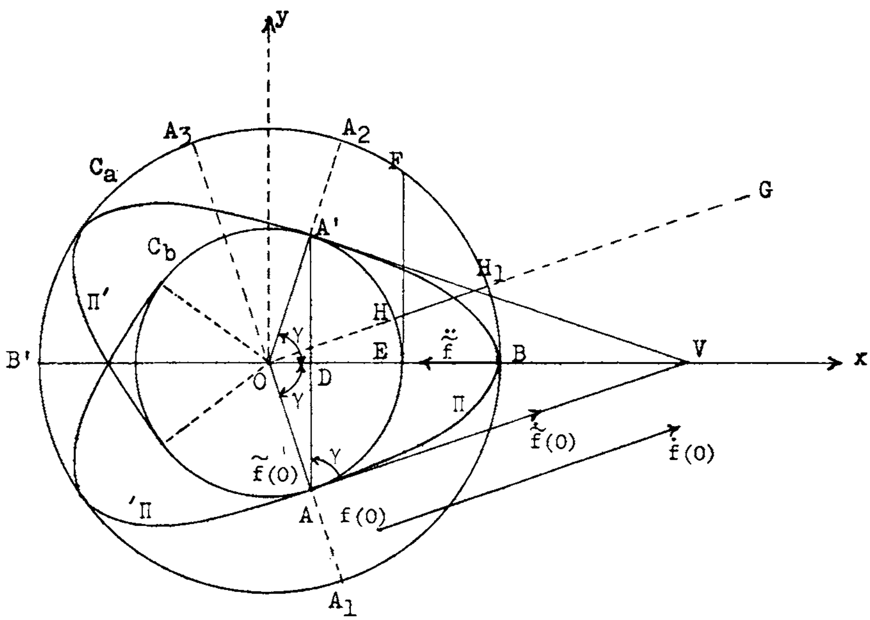

Fig.1

Fig. 1 shows the ring $R_{a,b}$ obtained by intersecting the solid shell with the plane $z=0$. We consider the parabolic arc $\Pi=\widehat{ABA'}$ having its vertex B on the circle C_a, and tangent to C_b at the points A, A'. It will aid our computations if we first show how Π can be constructed. Erect EF perpendicular to OB and intersecting C_a at F. Mark the two points D and V such that

(2) $\qquad DB=BV=EF,$

and erect the perpendicular to OB at D, intersecting C_b at A and A'. Join V to A and to A'. We claim that

(3) $\qquad \angle\, OAV=\angle\, OA'V=90°.$

PROOF: From the triangle OEF we find that $(EF)^2=a^2-b^2$. However, by (2)

$$OD=a-DB=a-EF,\ OV=a+BV=a+EF,$$

and so $OD\cdot OV=a^2-(EF)^2=a^2-(a^2-b^2)=b^2=(OA')^2$, or $OD\cdot OV=(OA')^2$. This is a characteristic relation for a right-angled triangle and establishes (3).

Incidentally, we have shown that in fig. 1 we have

(4) $\qquad OV=a+\sqrt{a^2-b^2}.$

From $DB=BV$ it also follows that the parabolic arc Π with vertex at B and passing through A and A', is also tangent to C_b at A and A'.

Next we consider on the arc $\Pi=\widehat{ABA'}$ a motion $\mathfrak{f}(t)$ $(\alpha\leqq t\leqq\beta)$, which we call *Galilean*; by this we mean that the point $\mathfrak{f}(t)$ describes the arc Π, $\mathfrak{f}(\alpha)=A$, $\mathfrak{f}(\beta)=A'$, and that the point $\mathfrak{f}(t)$ has a *constant acceleration* $\ddot{\mathfrak{f}}$ which must, of course, be real and negative. For this motion the modulus $|\dot{\mathfrak{f}}(t)|$ of the velocity vector assumes its maximal value at the endpoints A and A'. Let $v=|\dot{\mathfrak{f}}(\alpha)|=|\dot{\mathfrak{f}}(\beta)|$ denote this maximal value. Along Π we therefore have

(5) $\qquad F(\mathfrak{f})=v/\sqrt{|\ddot{\mathfrak{f}}|}.$

This functional being invariant if we change the scale in t, without loss of generality we may assume $\alpha=0$, $\beta=1$, and now

(6) $\qquad \mathfrak{f}(t),\ (0\leqq t\leqq 1)$

represents the Galilean motion along the arc Π.

The quantity $F(\mathfrak{f})$ is of dimension $LT^{-1}(LT^{-2})^{-1/2}=L^{1/2}$. Let us show that *in terms of fig.* 1 *we have*

(7) $\qquad F(\mathfrak{f})=\sqrt{OV}.$

PROOF: It is well known that a material particle projected vertically upward with the initial velocity v_0, will reach a maximal height h given by the relation

$$(8) \qquad v_0^2 = 2gh.$$

Using (5), the relation (7), to be established, amounts to

$$(9) \qquad v^2 = |\ddot{f}| \cdot OV.$$

The motion $f(t)$ may be regarded as a motion in the field of constant acceleration $\ddot{f} = -|\ddot{f}|$, while its initial "vertical" velocity is $v_0 = v \sin \gamma$, where $\gamma = \angle DAV$. By (8) we have

$$(v \sin \gamma)^2 = 2|\ddot{f}| \cdot DB = \ddot{f} \cdot DV,$$

in view of (2). It follows that

$$v^2 = |\ddot{f}| \frac{DV}{(\sin \gamma)^2} = |\ddot{f}| \cdot \frac{AV}{\sin \gamma} = |\ddot{f}| \cdot OV$$

proving (9).

We now *extend the motion* (6) *to all real* t *by the functional equation*

$$(10) \qquad f(t+1) = e^{2i\gamma} f(t) \text{ for all real } t.$$

This amounts to successive rigid rotations of the motion on Π through the angle $2\gamma = \angle AOA'$, and shows that the curve Γ described by the extended motion $f(t)$ $(t \in \mathbf{R})$, is an infinite sequence of parabolic arcs

$$\ldots {}''\Pi' \Pi \Pi \Pi' \Pi'' \ldots,$$

all congruent to Π. Therefore (10) defines a motion $f(t)$ having the properties

$$(11) \qquad f(t) \in (R_{a,b}),$$

$$(12) \qquad f(t) \in C^1(\mathbf{R}),$$

$$(13) \qquad F(f) = \sqrt{OV} = \sqrt{a + \sqrt{a^2 - b^2}},$$

the last by (7) and (4).

Notice that the acceleration vector $\ddot{f}(t)$ is piecewise constant, with discontinuities for all integer values of t, while for all other values of t, its modulus is constant, in fact

$$(14) \qquad |\ddot{f}(t)| = |\ddot{f}|.$$

After these extensive preliminaries, we can pass to *a proof of Theorem* 1. *Let* $f(t)$ *be a motion within the solid shell* (1) *subject to the condition*

$$(15) \qquad \|f\| \leqq \|\ddot{f}\| = -\ddot{f},$$

and we are to show that this implies that

$$\|\dot{f}\| \leqq \|\dot{\tilde{f}}\| = |\dot{\tilde{f}}(0)| = v. \tag{16}$$

By lemma 1 this would imply that $L(S) = F(\tilde{f})$, and prove Theorem 1 that

$$L(SoSh_{a,b}) = F(\tilde{f}) = \sqrt{a + \sqrt{a^2 - b^2}}. \tag{17}$$

To prove (16) we assume that for some t_0 we have $|\dot{f}(t_0)| > v$ and get a contradiction. We may as well assume that $t_0 = 0$, hence

$$|\dot{f}(0)| > v = |\dot{\tilde{f}}(0)|. \tag{18}$$

We begin by rotating the motion $f(t)$ rigidly about 0, hence within the shell S, so that the vector

$$\dot{f}(0) \textit{ is parallel to the vector } \dot{\tilde{f}}(0). \tag{19}$$

If $f(0)$ is in the half-space containing the arc Π and bounded by the vertical plane π that intersects the horizontal plane xOy in the line A_1OA_3, we do nothing further. However, if $f(O)$ is not in the half-space, then we reflect the entire motion $f(t)$ with respect to π, so as to have now $f(O)$ in that half-space; notice that (19) still holds after this reflection. Let $OG \perp A_1A_3$, hence $OG \perp \pi$. Now we rotate the motion $f(t)$ rigidly about the axis of rotation OG so that $f(O)$ falls with the quadrant A_1OG. This places $f(O)$ within the (partly) curvilinear quadrilateral

$$Q = AA_1H_1H. \tag{20}$$

Assuming that

$$2\gamma \geqq \frac{\pi}{2}, \tag{21}$$

which is the case of fig. 1, we find that

$$f(O) \in Q' = AA_1A_2A', \tag{22}$$

and it follows that

$$\operatorname{Re} f(O) \geqq OD. \tag{23}$$

Notice that (19) still holds. We may even assume that the vector

$$\dot{f}(O) \text{ is parallel and } \textit{points in the same direction as } \dot{\tilde{f}}(O). \tag{24}$$

For if not we replace $f(t)$ by $f(-t)$, thereby reversing the direction of $\dot{f}(O)$.

We now consider the motions $\tilde{f}_p(t)$ and $f_p(t)$ obtained by projecting orthogonally on Ox the motions $\tilde{f}(t)$ and $f(t)$, respectively. Clearly

$$\tilde{f}_p(t) = \operatorname{Re} \tilde{f}(t), \quad f_p(t) = \operatorname{Re} f(t). \tag{25}$$

Let us now compare these rectilinear motions during the time interval from $t=0$ to $t=\frac{1}{2}$. The motion $\tilde{f}_p(t)$ starts from D, at $t=0$, with initial velocity $\dot{\tilde{f}}_p(O)=v\sin\gamma$, and, while uniformly decelerated with constant deceleration $|\ddot{\tilde{f}}|$, just manages to arrive at B, at $t=\frac{1}{2}$, with *zero velocity.* The motion $f_p(t)$ starts at $\operatorname{Re} f(O)$, *ahead of* $\tilde{f}_p(O)$, because of (23), with a *larger initial velocity*, because of (24) and (18), and moves during $0\leqq t\leqq\frac{1}{2}$ with an acceleration $\ddot{f}_p(t)$ that nowhere exceeds in modulus the value $|\ddot{\tilde{f}}|$, because of (15). It follows that *the point $f_p(t)$ must have reached the point B, for some $t<\frac{1}{2}$, with a positive velocity.* This being cinematically impossible, we have obtained the desired contradiction.

There is a slight additional difficulty if (21) does not hold, but rather

$$2\gamma<\frac{\pi}{2}. \tag{26}$$

Indeed now we would have in fig. 1 that

$$Q=AA_1H_1H\supset Q'=AA_1A_2A'.$$

It follows that while $f(O)$ is in Q, $f(O)$ need not be in Q'. In this case, let δ denote the *positive* angle $\delta=\angle A'Of(O)$. For the last time we rotate the motion $f(t)$ rigidly *clockwise* about Oz by the angle δ. As a result the new position of $f(O)$ lies on the segment $[A', A_2]$, and our previous arguments will apply. Indeed observe that (23) will now hold, and that the value of $\dot{f}_p(O)$ has only increased by the last rotation.

REMARK. In fig. 1 we let $b\to 0+$. As a result we find that

$$\lim_{b\to 0+}\tilde{f}(t)=a\cdot\mathscr{E}_2(t),$$

uniformly in every finite interval of the t-axis: The extremizing motion for the solid shell converges to the extremizing motion of $I_a=[-a, a]$.

6. COROLLARIES OF THEOREMS 1 AND 2

It was already mentioned that the relations (2.1) and (2.4) are special examples of Theorem 2.

PROOF OF (2.9). From $R_{a,b}\subset SoSh_{a,b}$, the monotonicity property (2.2) and Theorem 1, we obtain that

$$L(R_{a,b})\leqq\sqrt{a+\sqrt{a^2-b^2}}.$$

On the other hand we know that $\tilde{f}(t)\in(R_{a,b})$, and therefore

$$F(\tilde{f})=\sqrt{a+\sqrt{a^2-b^2}}\leqq L(R_{a,b}).$$

PROOF OF (2.6). From $C_a \subset R_{a,b}$ and (2.9) we obtain that

$$L(C_a) \leqq \sqrt{a+\sqrt{a^2-b^2}}.$$

Letting $b \to a$ we obtain that

$$L(C_a) \leqq \sqrt{a}.$$

However, in section 2 we found that for $\tilde{f}(t) = ae^{it}$ we have $F(\tilde{f}) = \sqrt{a}$.

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